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Cardinal Series Interpolation to Nonuniform Grids

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We consider Whittaker's cardinal series interpolation from a uniform grid to a nonuniform grid. It is shown that the interpolation is a bounded linear map if and only if a constraint on the clustering of the nonuniform grid is satisfied. An error estimate showing the spectral accuracy of the cardinal series interpolation is also presented. © 1994 Academic Press, Inc.

1. INTRODUCTION

In this article we consider Whittakers's cardinal series interpolation between a uniform grid and a nonuniform grid. Cardinal series interpolation is a limiting case of cardinal spline interpolation [5] and is also a limit for interpolation by finite Fourier series as the extent of the grid increases. Because of this, cardinal series interpolation is important in the theory both for interpolation using Fourier methods and as a limiting case for more conventional local interpolation between finite difference or finite element grids. We are currently developing a more general theory of interpolation between grids in which the results proved here play a essential role.

Interpolation between grids is an important part of many computational procedures. Perhaps the most important current application arises in domain decomposition methods. Domain decomposition is used as a means to decompose large problems into smaller subproblems that can be solved in parallel; see the collections of papers [2, 1] for examples of the use of domain decomposition.

The primary result of this article is the determination of the necessary and sufficient condition for the cardinal series interpolation to be a

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bounded linear operator between the space of square summable sequences on the first grid and a weighted norm on the second grid. The principle technique used in the proofs is borrowed from the work of Paley and Wiener on almost periodic functions [4, p. 108].

In Section 2 we give the basic definitions, define the interpolation operator, and give the clustering constraint to be satisfied by the grid mapping to give a bounded operation. In Section 3 we show that the clustering constraint is necessary and sufficient for the one-dimensional cardinal series interpolation to be a bounded linear operator. In Section 4 we show the one-dimensional result can be extended to prove the result in higher dimensional space. Finally, error estimates for cardinal series interpolation are presented in Section 5.

2. Finite Difference Grids and Cardinal Series Interpolation

The usual orthogonal grid in R^d with grid spacing h, which we refer to as G, is

$$(hZ)^d = \{mh : m \in Z^d\},$$

where Z is the set of integers. We consider cardinal series interpolation from G to a second grid $\bar{G}$ defined as the image of $(hZ)^d$ under a mapping φ from R^d to R^d. Usually we assume that the mapping is independent of the grid spacing, although this requirement can be relaxed.

We take the same grid spacing h on both grids G and $\bar{G}$. Our treatment easily extends to using two different grid spacings h and $\hbar$, for G and $\bar{G}$, respectively, as long as the grid ratio $\hbar/h$ remains bounded above and bounded away from zero. For simplicity of exposition we use only the one parameter h.

The space of grid functions on G is the set of square summable sequences. That is, we consider sequences $\{u_m\}$ from $m \in Z^d$ and use the norm

$$\|u\|_h = \left(h^d \sum_{m \in Z^d} |u_m|^2 \right).$$

The Fourier transform is defined on this space by

$$\hat{u}(\xi) = \left(\frac{h}{\sqrt{2\pi}} \right)^d \sum_{m \in Z^d} e^{-imh \cdot \xi} u_m, \tag{2.1}$$

where ξ is restricted to the domain $B^d = [-\pi h^{-1}, \pi h^{-1}]^d$. The inversion formula is

$$u_m = \left(\frac{1}{\sqrt{2\pi}} \right)^d \int_{B^d} e^{imh \cdot \xi} \hat{u}(\xi) \, d\xi, \tag{2.2}$$

and Parseval's relation relating the grid function and transform is

$$h^d \sum_{m \in Z^d} |u_m|^2 = \int_{B^d} |\hat{u}(\xi)|^2 \, d\xi.$$

We define the norm of $\hat{u}$ as

$$\|\hat{u}\| = \left(\int_{B^d} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}.$$

For the grid $\bar{G}$ we consider a weighted L^2 pseudo-norm given by

$$\|v\|_\beta = \left(h^d \sum_{m \in Z^d} \beta_m \, |v_m|^2 \right)^{1/2},$$

where the weights β_m are nonnegative real numbers. We denote the space of sequences for which $\|\cdot\|_\beta$ is finite by $\mathscr{B}^d$. Note that $\|\cdot\|_\beta$ is a norm only if all the weights are positive, and this norm is equivalent to the L^2 norm in $\bar{G}$ only if the weights are bounded and bounded away from zero. Typically the weights β_m would be some approximation to the Jacobian of φ at the grid point mh.

The cardinal series interpolation of a function u defined on the grid G, which is hZ^d, is the function Φu on $\bar{G}$, given by

$$(\Phi u)_m = \left(\frac{1}{\sqrt{2\pi}} \right)^d \int_{B^d} e^{i\varphi(mh) \cdot \xi} \hat{u}(\xi) \, d\xi. \tag{2.3}$$

The discrete function Φu is not in $\mathscr{B}^d$ unless some conditions are imposed on φ. The necessary and sufficient condition is the clustering constraint.

The Clustering Constraint. The measurable function φ from R^d to R^d satisfies the clustering constraint on the grid $\bar{G}$ with weights β_m if there is a constant B and a positive number h_0 such that for $0 < h \leqslant h_0$ and any $v \in Z^d$ the relation

$$\sum_{m \in S_v} \beta_m \leqslant B$$

holds, where S_v is the set

$$\{m \colon |h^{-1}\varphi(mh) - v|_\infty \leqslant \tfrac{1}{2}\}. \tag{2.4}$$

The set S_v is the set of indices of points in $\bar{G}$ that are closest to vh. There are several important cases in which the clustering constraint is easily verified. In one dimension, if φ is a differentiable function and there are constants $\underline{b}$ and $\bar{b}$, such that $0 < \underline{b} \leqslant \beta_m \leqslant \bar{b}$, then the clustering constraint is satisfied if the derivative of φ is bounded away from zero. As a consequence, φ must be a monotone function. Another case is with φ being a piecewise linear function, not even necessarily continuous, where the slopes of the linear segments are bounded away from zero.

3. Cardinal Series Interpolation in One Dimension

In this section we consider the special case of cardinal series interpolation in one dimension. The analysis of cardinal series interpolation in higher dimensions utilizes the result for one dimension.

In one dimension, the Fourier transform given by (2.1) can be written

$$\hat{u}(\xi) = \frac{h}{\sqrt{2\pi}} \sum_{\nu=-\infty}^{\infty} e^{-i\nu h\xi} u_\nu, \tag{3.1}$$

where u_ν denotes $u(\nu h)$ and ξ is restricted to the interval $[-\pi h^{-1}, \pi h^{-1}]$. The inversion formula (2.2) is

$$u_\nu = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{i\nu h\xi} \hat{u}(\xi)\, d\xi.$$

The cardinal series interpolation operator (2.3) is

$$(\Phi u)_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{i\varphi(mh)\xi} \hat{u}(\xi)\, d\xi. \tag{3.2}$$

By using the series (3.1) this can be expressed as a series,

$$(\Phi u)_m = \sum_{\nu=-\infty}^{\infty} u_\nu \frac{\sin(h^{-1}\varphi(mh) - \nu)\pi}{(h^{-1}\varphi(mh) - \nu)\pi},$$

which is the classical representation of the cardinal series interpolation.

The classical cardinal series is an interpolant of a sequence of data $\{u_\nu\}$ in l^2 to a continuous function $f(x)$, such that $f(\nu) = u_\nu$ for all ν, and is defined by

$$f(x) = \sum_{\nu=-\infty}^{\infty} u_\nu \frac{\sin(x-\nu)\pi}{(x-\nu)\pi};$$

see the preface to [5]. Here we make the slight modification to consider the interpolant such that $f(h\nu) = u_\nu$; that is, we introduce the grid scaling and, rather than consider the continuous function, we are concerned with the sequence obtained by the evaluation of the cardinal series at the discrete set of points on the other grid. The cardinal series is the limit of piecewise polynomial spline interpolation (see lecture 9 of [5]) and is also connected to theory of discrete entire functions (see [3]).

THEOREM 3.1. *The cardinal series interpolation operator Φ, given by* (3.2), *is a bounded operator from $L^2(hZ)$ to $\mathscr{B}$ if and only if the function φ satisfies the clustering constraint.*

The operator Φ is a bounded operator only if there is a constant C, independent of h for $0<h\leqslant h_0$, such that

$$\|\Phi u\|_\beta \leqslant C\|u\|_h \tag{3.3}$$

for all u in $L^2(hZ)$.

Proof. We first show that the clustering constraint is a necessary condition for Φ to be a bounded operator. Consider the discrete functions $u^{(\alpha)}$ defined by

$$u_v^{(\alpha)} = \begin{cases} h^{-1/2} & \text{if} \quad v=\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\|u^{(\alpha)}\|$ is 1 for each α. The Fourier transform of $u^{(\alpha)}$ is

$$\hat{u}^{(\alpha)}(\xi) = \frac{h^{1/2}}{\sqrt{2\pi}} e^{-i\alpha h\xi}$$

and therefore

$$\begin{aligned}(\Phi u^{(\alpha)})_m &= h^{-1/2} \frac{\sin(h^{-1}\varphi(mh)-\alpha)\pi}{(h^{-1}\varphi(mh)-\alpha)\pi} \\ &= h^{-1/2}\operatorname{sinc}(h^{-1}\varphi(mh)-\alpha)\pi,\end{aligned}$$

where $\operatorname{sinc}(y)$ is defined as $\sin(y)/y$. The norm of $\Phi u^{(\alpha)}$ in $\mathscr{B}$ is

$$\sum_{m=-\infty}^{\infty} \beta_m \,|\operatorname{sinc}(h^{-1}\varphi(mh)-\alpha)\pi|^2. \tag{3.4}$$

The series (3.4) is greater than the sum over the terms with $m\in S_\alpha$, i.e.,

$$\begin{aligned}&\sum_{m=-\infty}^{\infty} \beta_m \,|\operatorname{sinc}(h^{-1}\varphi(mh)-\alpha)\pi|^2 \\ &\quad\geqslant \sum_{m\in S_\alpha} \beta_m \,|\operatorname{sinc}(h^{-1}\varphi(mh)-\alpha)\pi|^2 \geqslant \left(\frac{2}{\pi}\right)^2 \sum_{m\in S_\alpha} \beta_m.\end{aligned}$$

If Φ is a bounded operator, then the series (3.4) is bounded independently of α; hence the sums $\sum_{m \in S_\alpha} \beta_m$ are bounded. This is precisely the clustering constraint.

We next show that the clustering constraint is sufficient for Φ to be a bounded operator from $L^2(hZ)$ to $\mathscr{B}$. Let v denote Φu. By multiplying each side of Eq. (3.2) by $\beta_m \bar{v}_m$ and summing over all m, we have

$$h \sum_{m=-\infty}^{\infty} \beta_m |v_m|^2 = \int_{-\pi/h}^{\pi/h} \overline{w(\xi)}\, \hat{u}(\xi)\, d\xi, \tag{3.5}$$

where

$$w(\xi) = \frac{h}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \beta_m v_m e^{-i\varphi(mh)\xi}. \tag{3.6}$$

We then obtain from (3.5)

$$\|v\|_\beta^2 \leqslant \|w\|\, \|u\|_h,$$

where the norm of w is that of $L^2([-\pi h^{-1}, \pi h^{-1}])$. We next show that

$$\|w\| \leqslant C\, \|v\|_\beta \tag{3.7}$$

for some constant C independent of v, from which we obtain

$$\|v\|_\beta \leqslant C\, \|u\|_h,$$

which is (3.3) and shows that v is in $\mathscr{B}$

We now prove (3.7). The essential ideas of this proof are due to Paley and Wiener [4] and used by them in their study of almost periodic functions.

For each integer m, define the integer ν_m by $\nu_m - \frac{1}{2} \leqslant h^{-1}\varphi(mh) < \nu_m + \frac{1}{2}$; then for each integer ν, S_ν is the set $\{m\colon \nu_m = \nu\}$, which is the same as (2.4). Given a function v in $\mathscr{B}$ we have the function $w(\xi)$ by (3.6) and define $z(\xi)$ by

$$\begin{aligned} z(\xi) &= \frac{h}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \beta_m v_m e^{-i\nu_m h\xi} \\ &= \frac{h}{\sqrt{2\pi}} \sum_{\nu=-\infty}^{\infty} \left(\sum_{m \in S_\nu} \beta_m v_m \right) e^{-i\nu h\xi}. \end{aligned} \tag{3.8}$$

We show that w and z are well-defined functions in $L^2(-\pi h^{-1}, \pi h^{-1})$ if φ satisfies the clustering constraint.

By Parseval's relation and the clustering constraint we have

$$\int_{-\pi/h}^{\pi/h} |z(\xi)|^2 \,\delta\xi = h \sum_{\nu=-\infty}^{\infty} \left| \sum_{m \in S_\nu} \beta_m v_m \right|^2$$

$$\leqslant h \sum_{\nu=-\infty}^{\infty} \sum_{m \in S_\nu} \beta_m |v_m|^2 \sum_{m \in S_\nu} \beta_m \leqslant Bh \sum_{m=-\infty}^{\infty} \beta_m |v_m|^2.$$

Thus

$$\|z\| \leqslant B^{1/2} \|v\|_h, \tag{3.9}$$

and z is in $L^2(-\pi h^{-1}, \pi h^{-1})$.

We now consider the difference between $w(\xi)$ and $z(\xi)$, where we now let ξ range throughout R. We have

$$\begin{aligned} w(\xi) - z(\xi) &= \frac{h}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \beta_m v_m (e^{-i\varphi(mh)\xi} - e^{-i\nu_m h\xi}) \\ &= \frac{h}{\sqrt{2\pi}} i\xi \sum_{m=-\infty}^{\infty} \beta_m v_m \int_{\varphi(mh)}^{\nu_m h} e^{-it\xi}\,dt \\ &= \frac{i\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\xi} V(t)\,dt, \end{aligned} \tag{3.10}$$

where

$$V(t) = h \sum_{m=-\infty}^{\infty} \varepsilon_m(t)\, \beta_m v_m$$

and

$$\varepsilon_m(t) = \begin{cases} 1 & \text{if } \varphi(mh) \leqslant t < \nu_m h \\ -1 & \text{if } \nu_m h \leqslant t < \varphi(mh) \\ 0 & \text{otherwise.} \end{cases}$$

By the clustering constraint, the sum defining $V(t)$ is absolutely convergent for each value of t. By considering ν such that $t - \frac{1}{2}h \leqslant \nu h < t + \frac{1}{2}h$, we have

$$\begin{aligned} |V(t)|^2 &= \left| h \sum_{m \in S_\nu} \varepsilon_m(t)\, \beta_m v_m \right|^2 \\ &\leqslant h^2 \sum_{m \in S_\nu} \beta_m |\varepsilon_m(t)|^2 \sum_{m \in S_\nu} \beta_m |v_m|^2 \\ &\leqslant h^2 B \sum_{m \in S_\nu} \beta_m |v_m|^2. \end{aligned} \tag{3.11}$$

Formula (3.10) shows that the function $(i\xi)^{-1}\,(w(\xi)-z(\xi))$ is the Fourier transform on R of the function $V(t)$. By Parseval's relation on R we have

$$\int_{-\infty}^{\infty} \frac{|w(\xi)-z(\xi)|^2}{|\xi|^2}\,d\xi = \int_{-\infty}^{\infty} |V(t)|^2\,dt. \tag{3.12}$$

From the left-hand side of (3.12) we have

$$\frac{h^2}{\pi^2}\int_{-\pi/h}^{\pi/h} |w(\xi)-z(\xi)|^2\,d\xi \leqslant \int_{-\infty}^{\infty} \frac{|w(\xi)-z(\xi)|^2}{|\xi|^2}\,d\xi. \tag{3.13}$$

To estimate the right-hand side of (3.12) we use the estimate (3.11) for $V(t)$ and obtain

$$\begin{aligned}\int_{-\infty}^{\infty} |V(t)|^2\,dt &= \sum_{\nu=-\infty}^{\infty} \int_{(\nu-1/2)h}^{(\nu+1/2)h} |V(t)|^2\,dt \\ &\leqslant h^3 B \sum_{m=-\infty}^{\infty} \beta_m\,|v_m|^2 \\ &= h^2 B\,\|v\|_\beta^2. \end{aligned} \tag{3.14}$$

Combining the two estimates (3.13) and (3.14) for (3.12) we obtain

$$\|w-z\|^2 = \int_{-\pi/h}^{\pi/h} |w(\xi)-z(\xi)|^2\,d\xi \leqslant \pi^2 B\,\|v\|_\beta^2.$$

Together with (3.9) this implies

$$\|w\| \leqslant B^{1/2}(1+\pi)\,\|v\|_\beta,$$

which is essentially (3.7) and thus (3.3) holds. This proves Theorem 3.1.

4. Cardinal Series Interpolation in Higher Dimensions

We now prove the extension of Theorem 3.1 to higher dimensions.

THEOREM 4.1. *The cardinal series interpolation operator is a bounded operator from* $L^2(hZ)^d$ *to* $\mathscr{B}^d$ *if and only if the function* φ *satisfies the clustering constraint.*

As with the one-dimensional interpolation, it is easy to show that the clustering constraint is necessary for Φ to be a bounded linear functional. The proof that the clustering constraint is sufficient depends on proving that the function

$$w(\xi)=\left(\frac{h}{\sqrt{2\pi}}\right)^d \sum_{m\in Z^d} \beta_m v_m e^{-i\varphi(mh)\cdot\xi}$$

is bounded by the norm of v. This proof is made easier by employing a convenient notation to keep track of the many functions used in the proof. In place of the two functions z and $h^{-1}V$ used in the one-dimensional case we employ $2^{d+1}-2$ additional functions. We designate these as w_σ, where σ is an element of $\{0, 1\}^k$ for some k, $0\leqslant k\leqslant d$. We let w correspond to the null-sequence with k equal to 0. The proof proceeds by taking one dimension at a time and decomposing each of the functions w_σ into two functions as was done in the proof of Theorem 3.1

We define the functions $p_0(m, j, \eta)$ and $p_1(m, j, \eta)$ as

$$p_0(m, j, m)=e^{i\nu^j(m)h\eta}$$

and

$$p_1(m, j, \eta)=\begin{cases}1 & \text{if } \varphi^j(mh)\leqslant\eta<\nu^j(m)h\\ -1 & \text{if } \nu^j(m)h\leqslant\eta<\varphi^j(mh)\\ 0 & \text{otherwise,}\end{cases}$$

where $\nu^j(m)$ is defined by

$$\nu^j(m)-\tfrac{1}{2}\leqslant h^{-1}\varphi^j(mh)<\nu^j(m)+\tfrac{1}{2}.$$

The point $\nu(m)h$ is a point in $(hZ)^d$ that is nearest to $\varphi(mh)$.

The function w_σ, for a k-tuple σ in $\{0, 1\}^k$, is defined by

$$\begin{aligned}&w_\sigma(\eta_1 ..., \eta_k, \xi_{k+1}, ..., \xi_d)\\ &\quad=h^d\sum_{m\in Z^d}\beta_m v_m\prod_{j=1}^{k} p_{\sigma_j}(m, j, \eta_j)\exp\left(i\sum_{j=k+1}^{d}\varphi^j(mh)\xi_j\right).\end{aligned}\qquad(4.1)$$

If σ_j is 0, then the range of η_j is $[-\pi h^{-1}, \pi h^{-1}]$ and if σ_j is 1, then the range is R. The range of ξ_j is $[-\pi h^{-1}, \pi h^{-1}]$.

We let σ, 0 refer to the $(k+1)$-tuple formed by adding a 0 to the k-tuple σ and similarly for σ, 1. Similar to the one-dimensional case we have the relation

$$\begin{aligned}&w_\sigma(\eta_1, ..., \eta_k, \xi_{k+1}, ..., \xi_d)-w_{\sigma,0}(\eta_1, ..., \eta_k, \xi_{k+1}, ..., \xi_d)\\ &\quad=i\xi_{k+1}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\eta_{k+1}\xi_{k+1}}w_{\sigma,1}(\eta_1, ..., \eta_{k+1}, \xi_{k+2}, ..., \xi_d)\,d\eta_{k+1}\end{aligned}$$

and the resulting inequalities

$$\|w_\sigma\| \leqslant \|w_{\sigma,0}\| + \|w_\sigma - w_{\sigma,0}\| \tag{4.2}$$

and

$$\|w_\sigma - w_{\sigma,0}\| \leqslant C\,\|w_{\sigma,1}\|. \tag{4.3}$$

The relations (4.1) and (4.2) are established as was done in the proof of Theorem 3.1.

As a consequence of repeated application of (4.2) and (4.3) we obtain

$$\|w\| \leqslant C\Big(\sum_{\sigma \in \{0,1\}^d} \|w_\sigma\|\Big). \tag{4.4}$$

The proof that cardinal series interpolation is a bounded interpolation operator depends on showing that each term on the right-hand side of (4.4) is bounded by $\|v\|_\beta$.

We now consider a representative $\|w_\sigma\|$ from (4.4), where, without loss of generality, we may consider σ_j to be 0 for $1 \leqslant j \leqslant l$ and σ_j to be 1 for j larger than l. We have

$$\|w_\sigma\|^2 = \int_{\xi \in [-\pi h^{-1}, \pi h^{-1}]^l} \int_{t \in R^{d-l}} |w_\sigma(\xi_1, ..., \xi_l, t_{l+1}, ..., t_d)|^2\, dt\, d\xi$$

and

$$\begin{aligned} w_\sigma(\xi_1, ..., \xi_l, t_{l+1}, ..., t_d) &= \left(\frac{h}{\sqrt{2\pi}}\right)^l \sum_{m \in Z^d} \beta_m v_m e^{i\nu_m \cdot \xi h} \prod_{j=l+1}^{d} p_1(m, j, t_j) \\ &= \left(\frac{h}{\sqrt{2\pi}}\right)^l \sum_{\nu \in Z^l} e^{i\nu \cdot \xi h} \sum_{m \in S_{\nu,\mu(t)}} \beta_m v_m \prod_{j=l+1}^{d} p_1(m, j, t_j), \end{aligned} \tag{4.5}$$

where $\mu(t)$ is the $(d-l)$-tuple given by $t_j - \frac{1}{2}h \leqslant \mu_j h < t_j + \frac{1}{2}h$ and $S_{\nu,\mu(t)}$ is the set of grid indices m such that $\nu_m^j = \nu^j$ for $j = 1, ..., l$, and $\nu_m^j = \mu^j$ for $j = l+1, ..., d$.

By Parseval's relation in the first l arguments of w_σ we have

$$\|w_\sigma\|^2 = h^l \sum_{\nu \in Z^l} \int_{t \in R^{d-l}} \Big| \sum_{m \in S_{\nu,\mu(t)}} \beta_m v_m \prod_{j=l+1}^{d} p_1(m, j, t_j) \Big|^2 dt.$$

For each value of t in R^{d-l} and ν in Z^l, we have

$$\left|\sum_{m \in S_{\nu,\mu(t)}} \beta_m v_m \prod_{j=l+1}^{d} p_1(m, j, t_j)\right|^2 \leqslant \sum_{m \in S_{\nu,\mu(t)}} \beta_m |v_m|^2 \sum_{m \in S_{\nu,\mu(t)}} \beta_m \left|\prod_{j=l+1}^{d} p_1(m, j, t_j)\right|^2.$$

Since

$$\left|\prod_{j=l+1}^{d} p_1(m, j, t_j)\right|$$

is bounded by 1 for each value of $t \in R^{d-l}$ and $\nu \in Z^l$ we have, by the clustering constraint,

$$\left|\sum_{m \in S_{\nu,\mu(t)}} \beta_m v_m \prod_{j=l+1}^{d} p_1(m, j, t_j)\right|^2 \leqslant B \sum_{m \in S_{\nu,\mu(t)}} \beta_m |v_m|^2.$$

Thus

$$\|w_\sigma\|^2 \leqslant Bh^l \sum_{\nu \in Z^l} \int_{t \in R^{d-l}} \sum_{m \in S_{\nu,\mu(t)}} \beta_m |v_m|^2$$

$$= Bh^l \sum_{\nu \in Z^l} \sum_{\mu \in Z^{d-l}} \left(\int_{t \in B(\mu)} 1\right) \sum_{m \in S_{\nu,\mu}} \beta_m |v_m|^2,$$

where $B(\mu)$ is the region given by

$$(\mu_j - \tfrac{1}{2})h < t_j \leqslant (\mu_j + \tfrac{1}{2})h.$$

Since the volume of $B(\mu)$ is h^{d-l}, we have

$$\|w_\sigma\|^2 \leqslant Bh^d \sum_{\nu \in Z^l} \sum_{\mu \in Z^{d-l}} \sum_{m \in S_{\nu,\mu}} \beta_m |v_m|^2 = Bh^d \sum_{m \in Z^d} \beta_m |v_m|^2 = B \|v\|_\beta^2.$$

This proves Theorem 4.1.

5. Error Estimates

In this section we present estimates for the error incurred by cardinal series interpolation. We give proofs of the error estimates in the one-dimensional case; the proof for higher dimensions are sketched. We begin with a function u defined on R^d and consider the grid function Tu in $L^2(hZ)^d$ defined by

$$Tu_m = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{B^d} e^{imh \cdot \xi} \hat{u}(\xi)\, d\xi,$$

where B^d is as defined in (2.1). Note that the discrete transform of Tu is $\hat{u}$ restricted to B^d. We also consider the discrete function Eu_m obtained by evaluating u on the grid, i.e., $Eu_m = u(mh)$.

We consider the difference between the cardinal series interpolation of Tu, i.e., ΦTu, and u evaluated on the grid $\varphi(hZ)$, which we denote by $E_\varphi u$, i.e.,

$$E_\varphi u_m = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{R^d} e^{i\varphi(mh) \cdot \xi} \hat{u}(\xi)\, d\xi.$$

The interpolant of Tu is given by

$$\Phi Tu_m = \left(\frac{1}{\sqrt{2\pi}}\right)^d \int_{B^d} e^{i\varphi(mh) \cdot \xi} \hat{u}(\xi)\, d\xi,$$

and thus the error is given by

$$E_\varphi u_m - \Phi Tu_m = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{h|\xi|_\infty \geq \pi} e^{i\varphi(mh) \cdot \xi} \hat{u}(\xi)\, d\xi.$$

The expression $|\xi|_\infty$ refers to the supremum norm on R^d, i.e.,

$$|\xi|_\infty = \max_{1 \leq i \leq d} |\xi_i|.$$

The error, the difference between the function evaluated at the grid points and the interpolant from the evenly spaced points, is estimated using the next theorem.

THEOREM 5.1. *If φ satisfies the clustering constraint and $r \geq d$, then there is a constant C_r such that for $u \in H^r(R^d)$*

$$\|E_\varphi u - \Phi Tu\|_\beta \leq C_r h^r \left(\int_{h|\xi|_\infty \geq \pi} |\xi|^{2r}\, |\hat{u}(\xi)|^2\, d\xi \right)^{1/2}.$$

If we interpolate the function u evaluated at the grid points rather than Tu we obtain similar estimates.

COROLLARY 5.2. *If φ satisfies the clustering constraint and $r \geqslant d$, then there is a constant C_r such that for $u \in H^r(R^d)$*

$$\|E_\varphi u - \Phi E u\|_\beta \leqslant C_r h^r \left(\int_{h|\xi|_\infty \geqslant \pi} |\xi|^{2r} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}.$$

As a further consequence we have that if the grid function v is a good approximation to the smooth function u on the evenly spaced grid, then Φv is a good approximation to u on the unevenly spaced grid.

COROLLARY 5.3. *If φ satisfies the clustering constraint, $r \geqslant d$ and $u \in H^r(R^d)$, and v is a grid function with $\|v - Eu\|_h \leqslant \varepsilon$, then there is a constant C_r such that*

$$\|E_\varphi u - \Phi v\|_\beta \leqslant C_r \left(\varepsilon + h^r \left(\int_{h|\xi|_\infty \geqslant \pi} |\xi|^{2r} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2} \right).$$

Proof of Theorem 5.1. We first give the proof for the one-dimesional case; then we sketch the proof for higher dimensions. Let $v_m = E_\varphi u_m - \Phi T u_m$; then

$$\sum_{m=-\infty}^{\infty} \beta_m |v_m|^2 h = \int_{h|\xi| \geqslant \pi} \overline{w(\xi)}\, \hat{u}(\xi) \, d\xi$$
$$\leqslant \left(\int_{h|\xi| \geqslant \pi} |\xi|^{-2} |w(\xi)|^2 \, d\xi \right)^{1/2}$$
$$\times \left(\int_{h|\xi| \geqslant \pi} |\xi|^{2} |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}, \tag{5.1}$$

where $w(\xi)$ is defined as in (3.6).

As in Section 3, define $z(\xi)$ as in (3.8). We then have, by the triangle inequality,

$$\left(\int_{h|\xi| \geqslant \pi} |\xi|^{-2} |w(\xi)|^2 \, d\xi \right)^{1/2} \leqslant \left(\int_{h|\xi| \geqslant \pi} |\xi|^{-2} |z(\xi)|^2 \, d\xi \right)^{1/2}$$
$$+ \left(\int_{h|\xi| \geqslant \pi} |\xi|^{-2} |w(\xi) - z(\xi)|^2 \, d\xi \right)^{1/2}. \tag{5.2}$$

The first integral on the right-hand side of (5.2) is evaluated using the periodicity of $z(\xi)$,

$$\int_{h|\xi| \geqslant \pi} |\xi|^{-2} |z(\xi)|^2 \, d\xi = \int_{-\pi/h}^{\pi/h} |z(\xi)|^2 \sum_{n=-\infty}^{\infty}{}' |\xi + 2\pi h^{-1} n|^{-2} \, d\xi,$$

where the prime on the summation denotes the omission of the term for $n=0$. Now, since $|\xi| \leqslant \pi h^{-1}$

$$\sum_{n=-\infty}^{\infty}{}' \; |\xi + 2\pi h^{-1} n|^{-2} \leqslant 2 \frac{h^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{h^2}{3}.$$

So

$$\int_{h|\xi| \geqslant \pi} |\xi|^{-2} |z(\xi)|^2 \, d\xi \leqslant \frac{h^2}{3} \int_{-\pi/h}^{\pi/h} |z(\xi)|^2 \, d\xi \leqslant \frac{h^2}{3} B \, \|v\|_\beta^2$$

by (3.9).

The second integral on the right-hand side of (5.2) is estimated using the relations (3.10), (3.12), and the estimate (3.11). We obtain

$$\int_{h|\xi| \geqslant \pi} \frac{|w(\xi) - z(\xi)|^2}{|\xi|^2} \, d\xi \leqslant \int_{-\infty}^{\infty} |V(t)|^2 \, dt \leqslant h^2 B \, \|v\|_\beta^2. \tag{5.3}$$

Therefore,

$$\left(\int_{h|\xi| \geqslant \pi} |\xi|^{-2} \, |w(\xi)|^2 \, d\xi \right)^{1/2} \leqslant Ch \, \|v\|_\beta,$$

and so from (5.1)

$$\|E_\varphi u - \Phi T u\|_\beta \leqslant Ch \left(\int_{h|\xi| \geqslant \pi} |\xi|^2 \, |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}. \tag{5.4}$$

It easily follows from (5.4) that

$$\|E_\varphi u - \Phi T u\|_\beta \leqslant C\pi \left(\frac{h}{\pi} \right)^r \left(\int_{h|\xi| \geqslant \pi|} |\xi|^{2r} \, |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}$$

for any $r \geqslant 1$.

We now sketch the proof for higher dimensions. As in Section 4, we employ k-tuples σ to index the auxiliary functions. In place of Eq. (5.1) we have

$$\begin{aligned} \sum_{m \in Z^d} \beta_m \, |v_m|^2 \, h^d &= \int_{h|\xi|_\infty \geqslant \pi} \overline{w(\xi)} \, \hat{u}(\xi) \, d\xi \\ &\leqslant \left(\int_{h|\xi|_\infty \geqslant \pi} |\xi|_\infty^{-2d} \, |w(\xi)|^2 \, d\xi \right)^{1/2} \\ &\quad \times \left(\int_{h|\xi|_\infty \geqslant \pi} |\xi|_\infty^{2d} \, |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}. \end{aligned}$$

We define the function $\Psi(\xi, \sigma)$ for a k-tuple σ by

$$\Psi(\xi, \sigma) = \max\left(\frac{\pi}{h}, \{|\xi_i| : (i \leqslant k \text{ and } \sigma_i = 0) \text{ or } i > k\}\right).$$

The functions w_σ are defined as in (4.1), except here the range of ξ is R, and we include an extra factor of $h^{|\sigma|}$ in w_σ as defined by (4.5).

The first decomposition is

$$\left(\int_{h|\xi|_\infty \geqslant \pi} |\xi|_\infty^{-2d} |w(\xi)|^{2d} \, d\xi\right)^{1/2}$$
$$\leqslant \left(\int_{R^d} \frac{|w(\xi) - w_0(\xi)|^2}{\Psi(\xi, (1))^{2(d-1)} |\xi_1|^2} d\xi\right)^{1/2}$$
$$+ \left(\int_{h|\xi|_\infty \geqslant \pi} \frac{|w_0(\xi)|^2}{\Psi(\xi, (0))^{2d}} d\xi\right)^{1/2}$$
$$\leqslant C\left(\int_{R^d} \frac{|w_1(t_1, \xi_2, ..., \xi_d)|^2}{\Psi(\xi, (1))^{2(d-1)}} dt_1 \, d\xi_2 \cdots d\xi_d\right)^{1/2}$$
$$+ \left(\int_{h|\xi|_\infty \geqslant \pi} \frac{|w_0(\xi)|^2}{\Psi(\xi, (0))^{2d}} d\xi\right)^{1/2}.$$

Proceeding with a similar decomposition on the second dimension in both of the last integrals, we obtain at the kth step a sum of terms of the form

$$\left(\int_{R^d} \frac{|w_\sigma(\xi)|^2}{\Psi(\xi, \sigma)^{2(d-|\sigma|)}} d\xi\right)^{1/2}.$$

After all decompositions are performed, we have a sum of 2^d functions w_σ. As in Section 4, we consider the case where $\sigma_i = 0$ for $1 \leqslant i \leqslant l$ and $\sigma_i = 1$ for $l < i \leqslant d$. We have, by the $(2\pi/h)$-periodicity of w_σ in the first l dimensions,

$$\|w_\sigma\|^2 = \int_{\xi \in R^l} \int_{t \in R^{d-l}} \frac{|w_\sigma(\xi_1, ..., \xi_l, t_{l+1}, ..., t_d)|^2}{\Psi(\xi, \sigma)^{2l}} dt \, d\xi$$
$$= \int_{h|\xi|_\infty \leqslant \pi} \int_{t \in R^{d-l}} |w_\sigma(\xi_1, ..., \xi_l, t_{l+1}, ..., t_d)|^2$$
$$\times \sum_{n \in Z^l} \frac{1}{\psi(\xi + 2n\pi/h, \sigma)^{2l}} dt \, d\xi$$
$$\leqslant Ch^{2l} \int_{h|\xi|_\infty \leqslant \pi} \int_{t \in R^{d-l}} |w_\sigma(\xi_1, ..., \xi_l, t_{l+1}, ..., t_d)|^2 \, dt \, d\xi. \quad (5.5)$$

We next use Parseval's relation on the first l dimensions, and proceeding in a way similar to the proof of Theorem 4.1, we have that the last integral in (5.5) is equal to

$$Ch^{3l} \sum_{\nu \in Z^l} \int_{t \in R^{d-l}} \left| h^{d-l} \sum_{m \in S_{\nu,\mu(t)}} \beta_m v_m \prod_{j=l+1}^{d} p_1(m, j, t_j) \right|^2 dt$$

$$\leqslant BCh^{2d+l} \sum_{\nu \in Z^l} \int_{t \in R^{d-l}} \sum_{m \in S_{\nu,\mu(t)}} \beta_m |v_m|^2 \, dt$$

$$= BCh^{2d+l} \sum_{\nu \in Z^l} \sum_{\mu \in Z^{d-l}} \left(\int_{t \in B(\mu)} 1 \, dt \right) \sum_{m \in S_{\nu,\mu}} \beta_m |v_m|^2$$

$$= BCh^{3d} \sum_{\nu \in Z^d} \sum_{m \in S_\nu} \beta_m |v_m|^2 = C_2 h^{2d} \|v\|_\beta^2.$$

From this the theorem follows easily.

Proof of Corollary 5.2. We have, for one dimension,

$$Eu_m = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{imh\xi} \hat{u}(\xi) \, d\xi = u(mh)$$

and

$$Eu_m - Tu = \frac{1}{\sqrt{2\pi}} \int_{h|\xi| \geqslant \pi} e^{imh\xi} \hat{u}(\xi) \, d\xi.$$

It is then easily shown that

$$\|Eu_m - Tu\|_h \leqslant Ch \left(\int_{h|\xi| \geqslant \pi} |\xi|^2 \, |\hat{u}(\xi)|^2 \, d\xi \right)^{1/2}$$

similar to (5.4). We then have

$$\|E_\varphi u - \Phi Eu\|_\beta \leqslant \|E_\varphi u - \Phi Tu\|_\beta + \|\Phi(Tu - Eu)\|_\beta$$
$$\leqslant \|E_\varphi u - \Phi Tu\|_\beta + C \, \|Tu - Eu\|_h$$

by Theorem 3.1. The proof for more dimensions is similar.

Proof of Corollary 5.3. We have, for one dimension,

$$E_\varphi u - \Phi v\|_\beta \leqslant \|E_\varphi u - \Phi Eu\|_\beta + \|\Phi Eu - \Phi v\|_\beta$$
$$\leqslant \|E_\varphi u - \Phi Eu\|_\beta + C \, \|Eu - v\|_\beta.$$

The result then follows from Corollary 5.2.

6. Conclusions and Observations

The error estimates given here display the "infinite" order of accuracy of the cardinal series interpolation, that is, the order of accuracy depends only on the differentiability of the function being interpolated. Notice that there are no smoothness requirements on the grid mapping φ. A careful examination of the proofs shows that no use is made of the order of the points in the grid $\bar{G}$, and thus any rearrangement of how the points of G are mapped to the points in $\bar{G}$ is an equivalent mapping. In particular, the properties of the interpolation depend essentially on the location of the points in $\bar{G}$ and less on the mapping φ.

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